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2007 J. Phys. A: Math. Theor. 40 1835

(http://iopscience.iop.org/1751-8121/40/8/010)

IOP Publishing — Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **40** (2007) 1835–1844 doi:10.1088/1751-8113/40/8/010

Fermionic coherent states for pseudo-Hermitian two-level systems

O Cherbal[1], **M Drir**[1], **M Maamache**[2] **and D A Trifonov**[3]

[1] Physical Faculty, Theoretical Physics Lab, USTHB, BP 32 El-Alia, Bab Ezzouar, 16111 Algiers, Algeria

[2] Laboratoire de Physique Quantique et Systèmes Dynamiques, Department of Physics, Setif University, Setif 19000, Algeria

[3] Institute of Nuclear Research, 72 Tzarigradsko Chaussée, 1784 Sofia, Bulgaria

Received 27 August 2006, in final form 11 January 2007
Published 6 February 2007
Online at stacks.iop.org/JPhysA/40/1835

Abstract

We introduce creation and annihilation operators of pseudo-Hermitian fermions for two-level systems described by a pseudo-Hermitian Hamiltonian with real eigenvalues. This allows the generalization of the fermionic coherent states approach to such systems. Pseudo-fermionic coherent states are constructed as eigenstates of two pseudo-fermion annihilation operators. These coherent states form a bi-normal and bi-overcomplete system, and their evolution governed by the pseudo-Hermitian Hamiltonian is temporally stable. In terms of the introduced pseudo-fermion operators, the two-level system Hamiltonian takes a factorized form similar to that of a harmonic oscillator.

PACS numbers: 03.05.−w, 03.65.Ca, 03.65.Ge

1. Introduction

In the last few years, a great deal of interest has been devoted to the study of non-Hermitian Hamiltonians with real spectrum (see [1–17] and references therein). Bender and Boettcher were the first to touch on this issue [7], by introducing the notion of PT-symmetry for one-dimensional non-Hermitian Hamiltonian $H_\nu = p^2 + x^2(\mathrm{i}x)^\nu$, $(\nu \geqslant 0)$, that possesses real, positive and discrete spectrum. In [15, 16], the Bessis–Zinn Justin conjecture about the reality of the spectrum of the PT-symmetric Hamiltonian $-\mathrm{d}^2/\mathrm{d}x^2 - (\mathrm{i}x)^{2M}$ for $M \geqslant 1$ has been proven. A criterion for the reality of the spectrum of non-Hermitian PT-symmetric Hamiltonians is provided in [17].

By definition, a PT-symmetric Hamiltonians H satisfies the relation

$$[H, PT] = 0, \tag{1}$$

where P and T are the operators of parity and time-reversal transformations, respectively. These are defined according to

$$PxP = -x, \qquad PpP = TpT = -p, \qquad T\mathrm{i}1T = -\mathrm{i}1, \tag{2}$$

where x, p, 1 are the position, momentum, and identity operators, respectively, acting on the Hilbert space, and $\mathrm{i} := \sqrt{-1}$.

Later, Mostafazadeh [8–12] introduced the notion of pseudo-Hermiticity in order to establish the mathematical relation with the notion of PT-symmetry. He explored the basic structure responsible for the reality of the spectrum of non-Hermitian Hamiltonians and established that all the PT-symmetric non-Hermitian Hamiltonians are pseudo-Hermitian. He has also shown that any diagonalizable operators with discrete spectra is pseudo-Hermitian if and only if its eigenvalues are either real or grouped in complex-conjugate pairs (with the same multiplicity). Moreover, this result has been generalized to the class all PT-symmetric standard Hamiltonians having $\mathbb{R}$ as their configuration space and to the class of possibly nondiagonalizable Hamiltonians that admit a block-diagonalization with finite-dimensional diagonal blocks. In fact, many of the later developments in the field are anticipated in the paper by Scholtz, Geyer and Hahne [18] (see also [19]).

By definition [8], a Hamiltonian H is called pseudo-Hermitian if it satisfies the relation

$$H^{\dagger} = \eta H \eta^{-1}, \tag{3}$$

where η is a linear, Hermitian and invertible operator. One can also express the definition (3) in the form $H^{\#} = H$, where $H^{\#} = \eta^{-1} H^{\dagger} \eta$ is the η-pseudo-adjoint of H [8].

An interesting area where the pseudo-Hermiticity is applied is in the study of non-Hermitian two-level systems [11, 14]. These simple Hamiltonian systems accurately model many physical systems in condensed mater, atomic physics and quantum optics [20–26]. The latter field provides a beautiful implementation of the coherent states formalism [28–31]. Rabi oscillations in the non-Hermitian system of a two-level atom in electromagnetic field have been recently examined in [14]. In the preceding paper [32], we have shown how the exact evolution and nonadiabatic Hannay's angle of Grassmannian classical mechanics of spin one-half in a varying external magnetic field is associated with the evolution of Grassmannian invariant-angle coherent states.

In this paper, we extend the fermionic coherent states approach [33–35, 37] to two-level non-Hermitian Hamiltonians which are pseudo-Hermitian (*p-Hermitian*). The underlying number system is Grassmann algebra [38, 39]. The set of coherent states (CS) for pseudo-fermionic (shortly *p-fermionic*) system turned out to consist of two subsets of states, which are *bi-normalized and bi-overcomplete* (shortly bi-normal CS).

The paper is organized as follows. In section 2, we study a non-Hermitian two-level system (a two-level atom interacting with electromagnetic field) and its pseudo-Hermitian properties. Then, we introduce the creation and annihilation operators for the two-level p-Hermitian system with real energy spectrum, such that its Hamiltonian ascribes a form similar to that of the free harmonic oscillator: $H = \Omega(b^{\#}b - 1/2)$, where b and $b^{\#}$ are the pseudo-fermionic (p-fermionic) lowering and raising operators. In section 3, we construct the p-fermionic CS as eigenstates of two annihilation operators, the eigenvalues being complex Grassmann variables. The set of such eigenstates forms a bi-normal and bi-overcomplete system. Then, in section 4, we study the time evolution of the constructed p-fermionic CS for the corresponding two-level p-Hermitian system. This evolution is shown to be temporally stable. The paper ends with concluding remarks.

2. Two-level systems and pseudo-Hermitian fermions

We consider a two-level atom interacting with an electromagnetic field. The general state of the two-level atomic system is

$$|\psi\rangle = C'_a(t)|+\rangle + C'_b(t)|-\rangle,$$

where $C'_{a,b}$ are the amplitudes of being in $|\pm\rangle$. They are time dependent due to atom–field interaction. In the interaction picture (dipole interaction and phenomenologically described decay), and in the rotating wave approximation, the evolution of the system is described by the equation [24, 25]

$$\mathrm{i}\frac{\partial}{\partial t}\begin{pmatrix} C'_a(t) \\ C'_b(t) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -\mathrm{i}\gamma_a & \omega^* \\ \omega & -\mathrm{i}\gamma_b \end{pmatrix}\begin{pmatrix} C'_a(t) \\ C'_b(t) \end{pmatrix}. \tag{4}$$

The real constants γ_a, γ_b are the decay rates for the upper and lower levels, respectively. The quantity ω characterizes the radiation–atom interaction matrix element between the levels (ω^* is the complex conjugate). The basic vectors of the upper (lower) level are $|+\rangle$ and $|-\rangle$.

We remove the average effect of the decay terms by means of a nonunitary transformation in the state space,

$$|\psi\rangle \to U(t)|\psi\rangle, \qquad U(t) = \mathrm{e}^{\Gamma t}, \qquad \Gamma = \tfrac{1}{4}(\gamma_a + \gamma_b). \tag{5}$$

The probability amplitudes in the new representation are $C_i(t) = \exp(\Gamma t)C'_i(t)$, $i = a, b$, and the non-Hermitian Hamiltonian takes the following matrix form:

$$H = \frac{1}{2}\begin{pmatrix} -\mathrm{i}\delta & \omega^* \\ \omega & \mathrm{i}\delta \end{pmatrix}, \tag{6}$$

where $\delta = (\gamma_a - \gamma_b)/2$.

The trace of H, equation (6), is vanishing, and the determinant of H is real, $\det H = (\delta^2 - |\omega|^2)/4$. Therefore, it is η-pseudo-Hermitian (η-p-Hermitian) [10]. Its matrix is a particular case of a more general 2×2 traceless matrix studied in [11], where the complete biorthonormal system $\{|\psi_i\rangle, |\phi_i\rangle\}$ for H and the operator η are explicitly constructed. We reproduce this system and η (up to certain common factors) in our specific notation.

The eigenvalues E_i of $H, i = 1, 2$, and the related complete biorthonormal system are given by (we consider the nondegenerate case of $E_i \neq 0$)

$$E_1 = -\frac{\Omega}{2}, \qquad E_2 = \frac{\Omega}{2}, \tag{7}$$

$$|\psi_1\rangle = \frac{1}{\sqrt{2\Omega}}\begin{pmatrix} \frac{-\omega^*\sqrt{\Omega+\mathrm{i}\delta}}{|\omega|} \\ \sqrt{\Omega - \mathrm{i}\delta} \end{pmatrix}, \qquad |\psi_2\rangle = \frac{1}{\sqrt{2\Omega}}\begin{pmatrix} \frac{\omega^*\sqrt{\Omega-\mathrm{i}\delta}}{|\omega|} \\ \sqrt{\Omega + \mathrm{i}\delta} \end{pmatrix}, \tag{8}$$

$$|\phi_1\rangle = \frac{1}{\sqrt{2\Omega^*}}\begin{pmatrix} \frac{-\omega^*\sqrt{\Omega^*-\mathrm{i}\delta}}{|\omega|} \\ \sqrt{\Omega^* + \mathrm{i}\delta} \end{pmatrix}, \qquad |\phi_2\rangle = \frac{1}{\sqrt{2\Omega^*}}\begin{pmatrix} \frac{\omega^*\sqrt{\Omega^*+\mathrm{i}\delta}}{|\omega|} \\ \sqrt{\Omega^* - \mathrm{i}\delta} \end{pmatrix}, \tag{9}$$

where $\Omega = \sqrt{|\omega|^2 - \delta^2}$.

For both real and complex eigenvalues (i.e. real and complex Ω), the Hamiltonian (6) satisfies the p-Hermiticity relation (3) with η given by

$$\eta = \begin{pmatrix} 1 & \frac{\mathrm{i}\delta\omega^*}{|\omega|^2} \\ -\frac{\mathrm{i}\delta\omega}{|\omega|^2} & 1 \end{pmatrix}. \tag{10}$$

As noted in [14], the real eigenvalues correspond to the case where the dipole interaction is large relative to the damping effect (i.e. $|\omega|^2 > \delta^2$). In this case, the ordinary Rabi frequency is replaced by the 'pseudo-Rabi frequency' which in our notation are $|\omega|/2$ and $\Omega/2$ correspondingly.

In the case of $\Omega = 0$ (i.e. $|\omega|^2 = \delta^2$), the amplitudes C'_a, C'_b are given by [14] $(1 - \delta t)\exp(-\Gamma t)$ and $(\mathrm{i}\omega t/2)\exp(-\Gamma t)$ correspondingly, where $\Gamma = (\gamma_a + \gamma_b)/4$. Due to the exponential decay factor $\exp(-\Gamma t)$, any divergence [27] does not occur in our system.

In the case of $|\omega|^2 < \delta^2$ the eigenvalues of H are pure imaginary, but the Hamiltonian is still pseudo-Hermitian [14].

The PT-symmetry of our Hamiltonian (6) is considered in [14] following the method of Bender, Brody and Jones [4]. The Parity operator P of the two-level system can be defined as [4]

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

so that $P^2 = 1$, $P = P^{-1}$.

The generalized parity operator for the two-level systems is defined [12, 13] as $P = |\phi_1\rangle\langle\phi_1| - |\phi_2\rangle\langle\phi_2|$, which in our case results to

$$P = \begin{pmatrix} 0 & -\frac{\omega^*}{|\omega|} \\ -\frac{\omega}{|\omega|} & 0 \end{pmatrix}.$$

Different definitions have been introduced by Mostafazadeh [12] and Ahmed [13] for the antilinear time-reversal operator T. As in the paper [14], we use the representation introduced by Bender *et al* [4], namely,

$$T = K_0,$$

where K_0 is the complex conjugation operator. One has $K_0^2 = 1$, $(PT)^2 = 1$, and one finds that PT commutes with H, $PK_0HK_0^{-1}P^{-1} = H$. If one uses the generalized parity operator $P = |\phi_1\rangle\langle\phi_1| - |\phi_2\rangle\langle\phi_2|$, then one can take $T = UK_0$, where U is a 2×2 unitary diagonal matrix, $U_{22} = U_{11}^* = -\omega/|\omega|$.

The generalized charge conjugation operator C for two-level system is given by the expression [12] $C = |\psi_1\rangle\langle\phi_1| - |\psi_2\rangle\langle\phi_2|$, which in our case reads

$$C = \frac{1}{\Omega}\begin{pmatrix} \mathrm{i}\delta & -\omega^* \\ -\omega & -\mathrm{i}\delta \end{pmatrix} = -\frac{2}{\Omega}H.$$

From the last equation we deduce that C commutes with the Hamiltonian H, $[C, H] = 0$. This invariance property eliminates negative inner products [14].

Furthermore unless otherwise stated, we consider the small damping effect case of our system, i.e. $|\omega|^2 > \delta^2$, that is Ω real (and if real it is positive). One can verify that in this regime the operator η, equation (10), can be represented in terms of $|\phi_i\rangle$ as

$$\eta = \frac{\Omega}{|\omega|}(|\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2|) \equiv \frac{\Omega}{|\omega|}\eta_+. \tag{11}$$

The operator η is positive definite. The notation $\eta_+ = |\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2|$ was introduced by Mostafazadeh [40].

Now, let us introduce the annihilation operator b associated with the Hamiltonian H given in equation (6),

$$b = \frac{1}{2\Omega}\begin{pmatrix} -|\omega| & \frac{-\omega^*(\Omega+\mathrm{i}\delta)}{|\omega|} \\ \frac{\omega(\Omega-\mathrm{i}\delta)}{|\omega|} & |\omega| \end{pmatrix}. \tag{12}$$

Its adjoint operator reads (Ω is real)

$$b^+ = \frac{1}{2\Omega}\begin{pmatrix} -|\omega| & \frac{\omega^*(\Omega+\mathrm{i}\delta)}{|\omega|} \\ \frac{-\omega(\Omega-\mathrm{i}\delta)}{|\omega|} & |\omega| \end{pmatrix}, \tag{13}$$

and its η-p-Hermitian adjoint $b^{\#}$, defined by

$$b^{\#} = \eta^{-1} b^{+} \eta, \tag{14}$$

takes the form

$$b^{\#} = \frac{1}{2\Omega} \begin{pmatrix} -|\omega| & \frac{\omega^{*}(\Omega - \mathrm{i}\delta)}{|\omega|} \\ \frac{-\omega(\Omega+\mathrm{i}\delta)}{|\omega|} & |\omega| \end{pmatrix}. \tag{15}$$

Next, we examine the properties of the operators $b^{\#}$ and b. First is that $b^{\#}$ and b realize a pseudo-Hermitian generalization of the fermion algebra, namely,

$$b^{2} = b^{\#2} = 0, \qquad \{b, b^{\#}\} = bb^{\#} + b^{\#}b = 1. \tag{16}$$

$b^{\#}$ and b could be called the creation and annihilation operators of p-Hermitian fermions [41]. One can verify that they raise and lower the eigenvalues of H by a quantity $\Omega = 2E$, i.e. they act on the states $|\psi_i\rangle$ as follows:

$$b\,|\psi_1\rangle = 0, \qquad b\,|\psi_2\rangle = |\psi_1\rangle\,, \tag{17}$$

$$b^{\#}\,|\psi_2\rangle = 0, \qquad b^{\#}|\psi_1\rangle = |\psi_2\rangle, \tag{18}$$

The operator b annihilates the lowest eigenstates $|\psi_1\rangle$, and $b^{\#}$ brings this state onto the upper eigenstates $|\psi_2\rangle$.

Introducing the quadratic operator $N = b^{\#}b$, the p-fermionic number operator, we find the following natural anticommutation relations:

$$\{N, b\} = b \qquad \{N, b^{\#}\} = b^{\#}. \tag{19}$$

In terms of the operators b and $b^{\#}$, the Hamiltonian H is factorized (up to an additive C-number term) to a form similar to that of the free (boson) harmonic oscillator,

$$H = \Omega\left(b^{\#}b - \tfrac{1}{2}\right). \tag{20}$$

Taking the Hermitian conjugate of both sides of (20), we reaffirm the p-Hermiticity of H (according to definition (3)):

$$\begin{aligned} H^{+} &= \Omega\left(b^{+}\eta b \eta^{-1} - \tfrac{1}{2}\right) \\ &= \Omega\eta\eta^{-1}\left(b^{+}\eta b\eta^{-1} - \tfrac{1}{2}\right)\eta\eta^{-1} \\ &= \eta H \eta^{-1}. \end{aligned}$$

The above relations confirm that b and $b^{\#}$ are lowering and raising operators for the two-level p-Hermitian system (with real eigenvalues) and can be regarded as p-fermionic annihilation and creation operators. This is consistent with the limit $\delta = 0$, corresponding to a Hermitian Hamiltonian, when $\eta = 1$ and $b^{\#} = b^{+}$, i.e. the p-Hermitian generalization of the fermion algebra reduces to the usual fermion algebra. Quantum system with Hamiltonian of the form (20) should be referred to as *p-fermionic oscillator*.

3. Pseudo-fermionic coherent states

Having introduced the p-fermion lowering and raising operators, we now embark on the construction of the p-fermionic coherent states (CS) for our system described by p-Hermitian Hamiltonian H given in equation (6). We shall follow as closely as possible the scheme of fermionic CS developed in papers [33–35, 37], generalizing it to the p-fermion case. For the reader convenience, we begin with a brief reminder of the properties of complex Grassmann variables [33, 35–37], denoted here as ξ and ξ^*.

The complex Grassmannian variables ξ_i and their complex conjugates ξ_i^* satisfy the anticommutation relations:

$$\{\xi_i, \xi_j\} = \xi_i\xi_j + \xi_j\xi_i = 0, \tag{21}$$

$$\{\xi_i^*, \xi_j\} = 0, \qquad \{\xi_i^*, \xi_j^*\} = 0. \tag{22}$$

ξ_i's anticommute with b and $b^\#$,

$$\{\xi_i, b\} = 0, \qquad \{\xi_i^*, b\} = 0, \qquad \{\xi_i, b^\#\} = 0, \tag{23}$$

and have the following properties:

$$\xi|\psi_1\rangle = |\psi_1\rangle\xi, \qquad \xi|\psi_2\rangle = -|\psi_2\rangle, \tag{24}$$

$$\xi|\phi_1\rangle = |\phi_1\rangle\xi, \qquad \xi|\phi_2\rangle = -|\phi_2\rangle\xi. \tag{25}$$

The pseudo-Hermitian conjugation reverses the order of all fermionic quantities, both the operators and the Grassmann variables:

$$(b^\#\xi_i + \xi_i^* b)^\# = \xi_i^* b + b^\#\xi_i. \tag{26}$$

The Grassmann integration and differentiation over the complex Grassmann variables are given by

$$\int \mathrm{d}\xi\, 1 = 0, \qquad \int \mathrm{d}\xi\, \xi = 1, \qquad \int \mathrm{d}\xi^* 1 = 0, \qquad \int \mathrm{d}\xi^* \xi^* = 1, \tag{27}$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi} 1 = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}\xi}\xi = 1, \qquad \frac{\mathrm{d}}{\mathrm{d}\xi^*} 1 = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}\xi^*}\xi^* = 1. \tag{28}$$

The Grassmann integration of any function is equivalent to the left differentiation

$$\int \mathrm{d}\xi\; f(\xi) = \frac{\partial}{\partial\xi} f(\xi). \tag{29}$$

We define the displacement operators $D(\xi)$ for any set of complex Grassmannian variables ξ in the following way:

$$D(\xi) = \exp(b^\#\xi - \xi^* b) \tag{30}$$

$$= 1 + b^\#\xi - \xi^* b + \left(b^\# b - \tfrac{1}{2}\right)\xi^*\xi. \tag{31}$$

The pseudo-Hermitian adjoint $D^\#$ is given by

$$D^\#(\xi) = \exp\left(\xi^* b - b^\#\xi\right) \tag{32}$$

$$= 1 + \xi^* b - b^\#\xi + \left(b^\# b - \tfrac{1}{2}\right)\xi^*\xi. \tag{33}$$

These two operators satisfy the following displacement relation:

$$D^\#(\xi) b D(\xi) = b + \xi.$$

Using the explicit formulae of D and $D^\#$, and the anticommutation relations between operators $b, b^\#$ and Grassmann variable ξ, we establish that $D(\xi)$ are pseudo-unitary: $D^\#(\xi)D(\xi) = 1 = D(\xi)D^\#(\xi)$.

We now define the pseudo-fermionic coherent states (p-fermionic CS) $|\xi\rangle$ as eigenstates of the annihilation operator b,

$$b\,|\xi\rangle = \xi\,|\xi\rangle\,. \tag{34}$$

The eigenvalue ξ is a complex Grassmannian variable.

The Hermitian adjoint of the CS (the bra-vector) is $\langle\xi|$ and it is left eigenstate of $b^\dagger$, $\langle\xi|b^\dagger = \langle\xi|\xi^*$. In order to meet the alternative relation ${}_\eta\langle\xi|b^\# = {}_\eta\langle\xi|\xi^*$, one has to define ${}_\eta\langle\xi| \equiv (|\xi\rangle)^\# := \langle\xi|\eta$.

Similarly to the cases of Glauber bosonic CS [28] and of fermionic CS [33] our p-fermion eigenstates $|\xi\rangle$ can be constructed from the lowest (ground) eigenstate $|\psi_1\rangle$ of the Hamiltonian H, acting on it by the pseudo-unitary operator $D(\xi)$:

$$|\xi\rangle = D(\xi)\,|\psi_1\rangle\,. \tag{35}$$

By using the formula (31) for the displacement operator, we may write the state $|\xi\rangle$ in the form

$$|\xi\rangle = \exp\left(-\tfrac{1}{2}\xi^*\xi\right)(|\psi_1\rangle - \xi|\psi_2\rangle). \tag{36}$$

The Hermitian adjoint of the CS is

$$\langle\xi| = \langle\psi_1|\;|\,D^\dagger(\xi) \tag{37}$$

$$= \exp\left(-\tfrac{1}{2}\xi^*\xi\right)(\langle\psi_1| + \xi^*\langle\psi_2|), \tag{38}$$

and the inner product $\langle\xi|\xi\rangle$ is

$$\langle\xi|\xi\rangle = \langle\psi_1|\psi_1\rangle + (\langle\psi_2|\psi_2\rangle - \langle\psi_1|\psi_1\rangle)\xi^*\xi - 2\mathrm{i}\,\mathrm{Im}(\xi\langle\psi_1|\psi_2\rangle), \tag{39}$$

so that $|\xi\rangle$ are not normalized.

Now we have to examine the set $\{|\xi\rangle\}$ for (over)completeness. One can straightforwardly check (using the rules (21)–(27)) that neither the integral (against the measure $\mathrm{d}\xi^*\mathrm{d}\xi$) of the Hermitian $|\xi\rangle\langle\xi|$ nor the integral of p-Hermitian $|\xi\rangle_\eta\langle\xi|$ (unnormalized) projectors result in the identity operator:

$$\int \mathrm{d}\xi^*\,\mathrm{d}\xi\,|\xi\rangle\langle\xi| \neq 1, \qquad \int \mathrm{d}\xi^*\,\mathrm{d}\xi\,|\xi\rangle_\eta\langle\xi| \neq 1. \tag{40}$$

The way out of this impasse is suggested by the known transition from 'orthonormal system' of eigenstates of Hermitian H to the 'biorthonormal system' of states of p-Hermitian H. With this idea in mind, we introduce another continuous family of states, namely the eigenstates $\widetilde{|\xi\rangle}$ of the operator $\tilde{b}$, that annihilates the dual state $|\phi_1\rangle$,

$$\tilde{b}\widetilde{|\xi\rangle} = \xi\widetilde{|\xi\rangle}, \qquad \tilde{b} = \eta b\eta^{-1}. \tag{41}$$

Operator $\tilde{b}$ is nilpotent, $\tilde{b}^2 = 0$ and anticommutes with $b^\dagger$. Representing $b^\dagger = \eta\tilde{b}^\dagger\eta^{-1}$, we see that $b^\dagger$ is η'-p-Hermitian adjoint to $\tilde{b}$, $\eta' = \eta^{-1}$. Denoting this pseudo-conjugation by ${}^{\#\prime}$ we obtain the pair of p-fermionic operators $\tilde{b}$ and $\tilde{b}^{\#\prime}$,

$$\tilde{b}\tilde{b}^{\#\prime} + \tilde{b}^{\#\prime}\tilde{b} = 1, \qquad \tilde{b}^2 = (\tilde{b}^{\#\prime})^2 = 0. \tag{42}$$

In view of the p-fermionic algebra (42), we introduce new displacement operators

$$\widetilde{D}(\xi) = \exp(\tilde{b}^{\#\prime}\xi - \xi^*\tilde{b}), \qquad \widetilde{D}^{\#\prime}(\xi)\tilde{b}\widetilde{D}(\xi) = \tilde{b} + \xi,$$

and construct eigenstates of $\tilde{b}$ according to the above-described scheme (see equations (35), (36)),

$$\widetilde{|\xi\rangle} = \widetilde{D}(\xi)|\phi_1\rangle \tag{43}$$

$$= \exp\left(-\tfrac{1}{2}\xi^*\xi\right)(|\phi_1\rangle - \xi|\phi_2\rangle)\,. \tag{44}$$

The scalar product between $\widetilde{\langle\xi}|\widetilde{\xi\rangle}$ takes the form

$$\widetilde{\langle\xi}|\widetilde{\xi\rangle} = \langle\phi_1|\phi_1\rangle + (\langle\phi_2|\phi_2\rangle - \langle\phi_1|\phi_1\rangle)\xi^*\xi - 2i\,\mathrm{Im}(\xi\langle\phi_1|\phi_2\rangle), \tag{45}$$

while

$$\widetilde{\langle\xi}|\xi\rangle = \frac{|\omega|}{\Omega}\langle\xi|\eta|\xi\rangle = 1,$$

or, more generally,

$$\langle\xi|\widetilde{\zeta\rangle} = \langle\psi_1|D^\dagger(\xi)\widetilde{D}(\zeta)|\phi_1\rangle = \xi^*\zeta + \tfrac{1}{4}(2-\xi^*\xi)(2-\zeta^*\zeta). \tag{46}$$

By means of the two types of states $|\xi\rangle$ and $\widetilde{|\xi\rangle}$ the resolution of the identity is realized in the following way:

$$1 = \int \mathrm{d}\xi^*\,\mathrm{d}\xi\,|\xi\rangle\widetilde{\langle\xi|} = \int \mathrm{d}\xi^*\,\mathrm{d}\xi\,\widetilde{|\xi\rangle}\langle\xi|. \tag{47}$$

Equations (47) can be easily verified using the expansions of $|\xi\rangle$ and $\widetilde{|\xi\rangle}$ in terms of $|\psi_i\rangle$ and $|\phi_i\rangle$ (equations (36) and (44)) and the rules of permutation and integration (21)–(27).

We have obtained that the system of one-mode *p-fermionic CS* consists of two subsets, namely $\{|\xi\rangle\}$ and $\{\widetilde{|\xi\rangle}\}$. In view of (45) and (46) this continuous system should be called *bi-normalized and bi-overcomplete* or shortly system of *bi-normal CS*. Similarly, the two sets of pseudo-unitary operators $D(\xi)$, $\widetilde{D}(\xi)$ should be called *bi-unitary*:

$$D(\xi)\widetilde{D}^\dagger(\xi) = 1 = \widetilde{D}^\dagger(\xi)D(\xi).$$

Note that $D(\xi)$ is η-pseudo-unitary, while $\widetilde{D}(\xi)$ is η'-pseudo-unitary with $\eta' = \eta^{-1}$.

4. Time evolution of p-fermionic coherent states

A given parametric set of states is said to be realizable for a physical system if the time evolution $|\psi;t\rangle$ of any initial state $|\psi\rangle$ from the set, governed by the Hamiltonian, leaves the state in the set [42]. In other words $|\psi;t\rangle$, for any t, obeys the defining criteria of the set. In such a case one shortly says that the time evolution (of the parametric set of states) is *stable* [42]. In Hermitian mechanics, this means that the time dependence of the states is included, up to a phase factor, in the state parameters. For example, the time evolution $|\alpha;t\rangle$ of Glauber CS $|\alpha\rangle$ [28] is stable with respect to the harmonic oscillator evolution operator $\exp(-\mathrm{i}Ht)$, $H = \omega(a^\dagger a + 1/2)$:

$$|\alpha;t\rangle = \mathrm{e}^{-\mathrm{i}\omega t/2}|\alpha(t)\rangle, \qquad a|\alpha;t\rangle = \alpha(t)|\alpha(t)\rangle, \qquad \alpha(t) = \alpha\,\mathrm{e}^{-\mathrm{i}\omega t}. \tag{48}$$

In the case of our p-fermionic CS $\{|\xi\rangle, \widetilde{|\xi\rangle}\}$, the set parameter is the complex Grassmann variable ξ, the eigenvalue of the p-fermionic lowering operators b or $\tilde{b}$. The time evolution is stable if the evolved states $|\xi;t\rangle$ and $\widetilde{|\xi;t\rangle}$ remain eigenstates of the operators b and $\tilde{b}$, respectively,

$$b|\xi;t\rangle = \xi(t)|\xi;t\rangle, \qquad \tilde{b}\widetilde{|\xi;t\rangle} = \xi(t)\widetilde{|\xi;t\rangle}. \tag{49}$$

This implies that the time-evolved CS $|\xi;t\rangle$ and $\widetilde{|\xi;t\rangle}$ should form bi-normal and bi-overcomplete system.

Let us first consider the time evolution of an initial CS $|\xi\rangle$. Clearly, we have $|\xi;t\rangle = \exp(-\mathrm{i}Ht)|\xi\rangle$, $|\xi;0\rangle \equiv |\xi\rangle$. Using the form (36) of $|\xi\rangle$ and the facts that $|\psi_{1,2}\rangle$ are eigenstates of H (with eigenvalues $E_{1,2}$), we get

$$|\xi;t\rangle = \mathrm{e}^{-\mathrm{i}E_1 t}\left(1 - \tfrac{1}{2}\xi^*\xi\right)|\psi_1\rangle - \mathrm{e}^{-\mathrm{i}E_2 t}\xi|\psi_2\rangle. \tag{50}$$

Taking into account that $E_1 = -\Omega/2 \equiv -E$ and $E_2 = \Omega/2 \equiv E$, we put $\xi(t) = \mathrm{e}^{-\mathrm{i}2Et}\xi$ and rewrite the last equation in the form

$$|\xi;t\rangle = \mathrm{e}^{\mathrm{i}Et}\left[\left(1-\tfrac{1}{2}\xi(t)^*\xi(t)\right)|\psi_1\rangle - \xi(t)|\psi_2\rangle\right] = \mathrm{e}^{\mathrm{i}Et}|\xi(t)\rangle. \tag{51}$$

which manifests the stability of the time evolution of CS $|\xi\rangle$. Note that the overall time-dependent factor $\exp(\mathrm{i}Et)$ is a phase factor since E_i are real.

In a similar manner, we establish that the time evolution $\widetilde{|\xi;t\rangle}$ of an initial $\widetilde{|\xi\rangle}$ is stable (remains eigenstate of $\tilde{b}$):

$$\begin{aligned}\widetilde{|\xi;t\rangle} &= \exp(-\mathrm{i}H^\dagger t)\widetilde{|\xi\rangle}\\ &= \exp(-\mathrm{i}E_1 t)\left(1-\tfrac{1}{2}\xi^*\xi\right)|\phi_1\rangle - \exp(-\mathrm{i}E_2 t)\xi|\phi_2\rangle\\ &= \exp(\mathrm{i}Et)\left(\left(1-\tfrac{1}{2}\xi(t)^*\xi(t)\right)|\phi_1\rangle - \xi(t)|\phi_2\rangle\right) = \exp(\mathrm{i}Et)\widetilde{|\xi(t)\rangle}.\end{aligned} \tag{52}$$

The results (51) and (52) reveal the bi-normality and bi-overcompleteness of the family of time-evolved states $\{|\xi;t\rangle, \widetilde{|\xi;t\rangle}\}$ of the p-fermionic oscillator system (20): one has $\langle t;\xi\widetilde{|\xi;t\rangle} = 1$, and

$$1 = \int \mathrm{d}\xi^*\,\mathrm{d}\xi\,|\xi;t\rangle\widetilde{\langle t;\xi|} = \int \mathrm{d}\xi^*\,\mathrm{d}\xi\,\widetilde{|\xi;t\rangle}\langle t;\xi|. \tag{53}$$

We observe that here the time-evolved states $|\xi;t\rangle$ and $\widetilde{|\xi;t\rangle}$ differ from CS $|\xi(t)\rangle$ and $\widetilde{|\xi(t)\rangle}$ in phase factors only. In more general cases, the overall factors $\mathcal{N}(t)$ and $\widetilde{\mathcal{N}}(t)$ ascribed in the stable evolution of bi-normal and bi-overcomplete system of states could not be phase factors, but their product should equal unity, $\mathcal{N}^*(t)\widetilde{\mathcal{N}}(t) = 1$.

Finally, we have to note that a complementary bi-normal and bi-overcomplete system of states can be constructed, in a symmetrical manner using the operators $b^\#$ and $\widetilde{b}^\#$, that annihilate the 'upper level states' $|\psi_2\rangle$ and $|\phi_2\rangle$.

5. Concluding remarks

In this paper, we have generalized the fermionic coherent states (CS) for two-level systems described by pseudo-Hermitian Hamiltonian with real spectrum. Unlike the standard bosonic and fermionic cases the system of pseudo-fermionic (p-fermionic) CS consists of two subsets, which are bi-normalized and bi-overcomplete. In this sense, the system of p-fermionic CS can be regarded as a continuous analogue of the biorthonormal system of discrete eigenstates of p-Hermitian H. The two subsets are built up as eigenstates of the p-fermion annihilation operators b and $\tilde{b} = \eta b \eta^{-1}$, where η is the Hermitian operator that ensures the p-Hermiticity of the Hamiltonian, $H = \eta^{-1}H^\dagger\eta$. In terms of b and $b^\# = \eta^{-1}b^\dagger\eta$, the Hamiltonian is factorized to the form of p-fermionic oscillator, equation (20).

The evolution of the p-fermionic CS governed by the p-Hermitian two-level Hamiltonian (6) is shown to be time stable—the evolved states remain eigenstates of the p-fermionic annihilation operators, preserving the bi-normality and bi-overcompleteness of the system at later time. In the Hermitian limit of $\eta = 1$ (that is $\delta = 0$ in (6)), our p-fermionic CS and all related formulae recover standard fermionic CS of [33, 34]. Time evolution of fermionic CS for a Pauli spin in a slowly varying magnetic field was examined by Abe [35] (see also the comment [36]).

Acknowledgments

One of the authors (OC) acknowledges Drs M Le Bellac and J P Provost for their useful comments and valuable discussions.

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